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ABSTRACT

In response to theories of learning that attempt to get rid of contents, two arguments are provided which suggest that it is theoretically disputable that knowledge develops along the same kind of process for biology and history, physics and mathematics, or the geometry of the triangle, and the geometry of space. The first is that empirical studies show that the biggest difficulties met by students depend heavily on the contents of situations to be mastered. The second is that the search for general theories misses a very important epistemological point, namely, that concepts and competencies are solutions to specific problems. Two conceptual fields (additive and multiplicative structures) and the nature of a content-oriented approach concerning natural language are discussed and illustrated, which leads to a comprehensive definition of the nature of a concept. It is a triplet of three sets: the set of situations that make the concept meaningful; the set of invariants (theorems-in-action) that characterize the variety of competencies of students (these invariants are properties of the concept); and the set of symbolic representations that can be used to represent these properties and the situations. The importance of these three sets, the nature of the conceptual field, and the importance of a careful and profound analysis of the contents of knowledge are addressed. (JN)

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DIDACTICS AS A CONTENT-ORIENTED APPROACH TO RESEARCH ON THE LEARNING OF PHYSICS, MATHEMATICS AND NATURAL LANGUAGE

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The title I have given to this conference might be misleading if one understands it as a limitation of the scientific research field covered by didactics to the study of learning. The study of teaching is also in the field, as the object of didactics is the process of transmission and appropriation of knowledge.

But teaching relies more upon conceptions of learning than the reverse. So we have to start with learning.

The important idea in the title is of course contained in the expression "content-oriented approach" and my main purpose is to illustrate that idea.

It is somewhat trivial to say that learning depends on the contents of knowledge to be learned. But we know that many theories of learning have tried to get rid of the contents, with the aim of reaching the stage of general theories. This is the case of many structural theories as well as of the associationistic theories. Beyond the fact that general theories fail to help teachers in understanding the difficulties met by students for specific concepts and specific competences, it is theoretically disputable that knowledge develops along the same kind of process for biology and history, physics and mathematics, or even the geometry of the triangle, and the geometry of space.

I have two main arguments for this:

The first one is that empirical studies show, even in the limited domain of mathematics and physics, that the biggest difficulties met by students depend heavily on the contents of the situations to be mastered. I will illustrate this point later.

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The second one is that the search for general theories misses a very important epistemological point, namely that concepts and competences are solutions to specific problems that human beings have been faced with at one time or another. In other words, beyond the general consideration that knowledge always consists of concepts and competences, based upon properties that can be expressed in the same natural language or the same symbolic shape (graphs, tables, equations, etc.) we must not miss the essential point that every piece of knowledge refers to situations to be mastered or problems to be solved. Consequently, a developmental approach of knowledge and learning requires definitions that should enable us to deal with the situations for which a concept is meaningful. We need a theory of reference that refers concepts to situations. And, to apply this idea immediately, instead of giving definitions right now, I will start with examples.

1. Additive and multiplicative structures

An important part of my research work concerns the development, the learning and the teaching of the four operations: addition, subtraction, multiplication, division. But trying to understand the reasons why the complexity of so-called elementary-arithmetic problems is so varied, even when the arithmetical operation to be used is the same, has suggested to me the idea that the first step was to classify the situations in which one has to add, subtract, multiply or divide. This has led me to define two conceptual fields: additive structures and multiplicative structures.

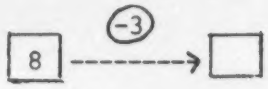
By additive structures I mean the set of situations whose handling requires one addition or one subtraction, or a combination of such operations.

By multiplicative structures I mean something similar for multiplication and division.

Additive and multiplicative structures are not independent of each other, but for the time being it is better to consider them separately. These definitions seem straightforward. But the outcome is not. The classification of addition and subtraction problems for instance results in a wide variety of relationships whose analysis requires different concepts like the concepts of measure, of state and transformation, of order relationship, of combination of measures, combination of transformations and combination of relationships, of natural number and directed number, of binary and unary operation, of directed state, abscissa and algebraic value, and others.

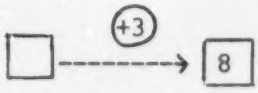
The reasoning (or relational calculus) that students have to make so as to find the right arithmetical operation in each case is so different that children usually fail to recognize that it is the same kind of problem, even when the arithmetic operation is the same. For instance:

Example 1: Susan has 8 dollars. She spends 3 dollars on cakes. How much does she have now?



find the final state

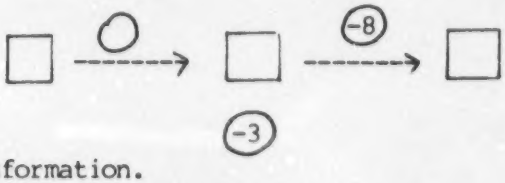
Example 2: Peter has just played a game of marbles. He has won 3 marbles and he has now 8 marbles. How many marbles did he have before playing?



find the initial state

Inverting the direct transformation into a subtraction is not an easy operation of thinking for 7-8 year-olds. It even conflicts with their primitive conception of subtraction as a decrease.

Example 3: Robert has played two games of marbles. He has lost 8 marbles in the second game, but he does not remember the first game. When he counts his marbles in the end, he finds 3 marbles less than what he had before playing the first game. What happened during the first game?



Find the first transformation.

There are two main difficulties in this case: you need reasoning without any information on the initial or final states; and you must subtract the whole loss from the second loss: the whole from the part, which is counter-intuitive.

I will leave additive structures for a while because I would like to pay more attention to multiplicative structures.

Nearly all problems involving a multiplication, a division or a combination of such operations have to do with proportion: either simple proportion between two variables, or multiple proportion of one variable to two or more independent variables.

Example 4: Eric wants to buy 4 miniature cars. They cost 3 dollars each. How much will he have to pay?

can be analyzed in three different ways:

- binary operation $4 \times 3 = \square$

which is fair enough if you have pure numbers in mind; but if you think of 4 cars and 3 dollars, you can't explain why multiplying cars by dollars gives dollars and not cars.

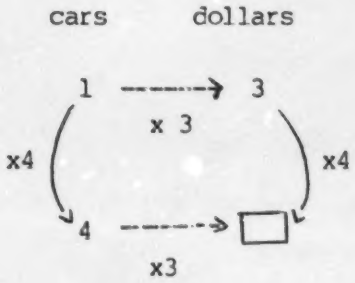
- unary operation: there are two possibilities

$$3 \xrightarrow{\times 4} \square$$

or

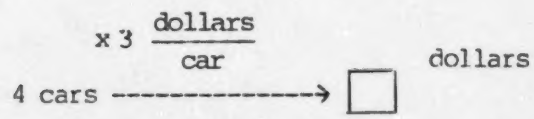
$$4 \xrightarrow{\times 3} \square$$

But the diagram of proportion between quantities of cars and quantities of dollars shows a strong conceptual difference between both possibilities.



x4 appears as a scalar operator that applies to 3 dollars and gives dollars. x4 expresses the relationship between two quantities of the same kind (4 cars and 1 car). It can be expressed in natural language by "4 times more", which is purely scalar.

x3 appears as a function operator that applies to 4 cars and gives dollars; x3 consists of a quotient of quantities "dollars per car". It cannot be expressed in natural language by "3 times more" and dimensional analysis can be used to show teachers (not children) the complete meaning.



If you need more details to be convinced of the difference, try to invert these operators.

"four times more" becomes "four times less"
but what about "three dollars per car"?

Let us come to the next example:

Example 5: What is the area of a bedroom which is 4 meters long and three meters wide.

The validity of the formula A = L x W relies upon four reasons.

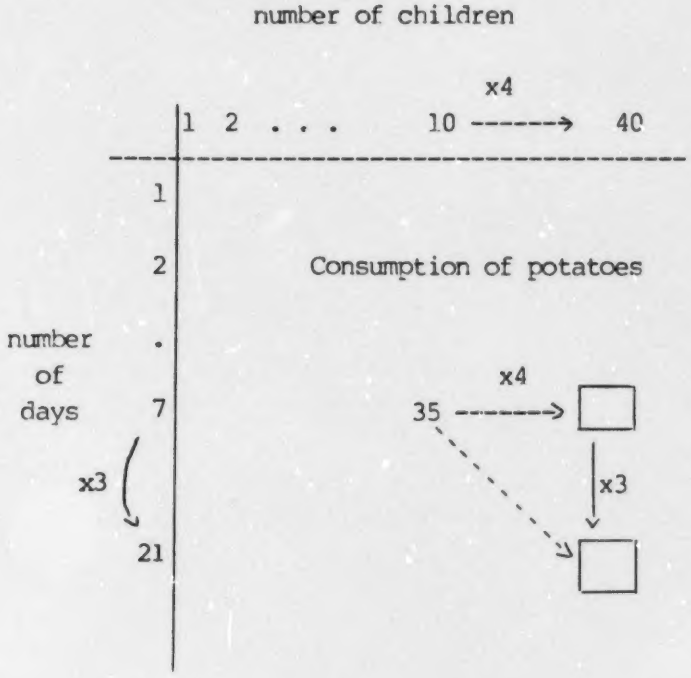
- Area is proportional to length when width is held constant;
- Area is proportional to width when length is held constant;
- Length and width are independent variables;
- The choice of units is such that the unit of area is the area of a rectangle of length 1 and width 1 (square).

So when you calculate the area of the bedroom, the multiplication 4 x 3 = □ is somewhat different from the one in example 4. It expresses the double proportion of area to length and width and a fair choice of the unit.

But suppose you consider the next example:

Example 6: 40 children go to a holiday camp. They will stay 21 days. The average consumption of potatoes for 10 children is 35 kilogrammes per week. What quantity of potatoes will they eat during their stay?

I have represented in a table the double proportion of the consumption of potatoes to the time and to the number of children; and one of the possible solutions of the problem: 4 times more for 40 children, and then 3 times more for 3 weeks.



I want to draw your attention just to one point.

- You can go directly from 35 kilogrammes to the unknown quantity of potatoes by multiplying by 12.
- 4 times more and then 3 times more is 12 times more.

This is an essential property of the bilinear function: consumption of potatoes as a function of the number of children and the number of days.

Whereas in the above example concerning the price of cars,

- the scalar procedure uses the isomorphic property of the linear function.

$$f(\lambda x) = \lambda f(x)$$

Price of 4 cars = 4 times the price of 1 car

$$x = 1 \qquad \lambda = 4 \qquad f(x) = 3$$

- and the function procedure uses the constant coefficient property of the linear function.

$$f(x) = a\,x$$

Price of 4 cars = 3 dollars per car x 4 cars

$$x = 4 \qquad a = 3$$

We now use, in the potatoe-consumption example, a property of the bilinear function

$$f(\lambda_1 x_1, \lambda_2 x_2) = \lambda_1 x \lambda_2 f(x_1, x_2)$$

$$f(4 \times 10, 3 \times 7) = 4 \times 3\, f(10, 7)$$

The consumption of 4 times 10 children during 3 times 7 days, is 12 times (4 x 3) the consumption of 10 children during 7 days.

Of course students do not know explicitely these axioms and theorems of linearity, but they do use them in solving problems. I call them "theorems in action", and one can trace the use of theorems in action both in additive structures and multiplicative structures. There is a large variety of such theorems. Some of them are discovered or understood at a very young stage of cognitive development (by 3 or 4 year-olds for some local properties of addition and subtraction). Some are still difficult for most 16 or 17 year-olds and adults. If you consider the different kinds of multiplication that I have just described, it is clear that the multiplication 4 x 3 = 12 cannot give account of the conceptual difficulty of the different problems presented.

You will not be surprised if I say that more than 50% of 15-16 year-old students fail to solve the following problem:.

Example 7: "Mr. Dupont has a small aquarium in his kitchen, and a large one in his den. The den one is twice as long, three times as wide, and twice as deep as the kitchen one. How many times is the den one larger than the kitchen one?"

And still in that case, it is possible to find the solution by paving mentally the den aquarium with the kitchen one, taken as a unit: two in the length, three in the width and so on. What would have been the success rate if I had said 1,8 times as long and 2,3 times as wide?

The concept of volume is an interesting example. It is both a physical and a mathematical concept. Its mathematization involves geometry and several branches of mathematics. I will consider here only the problem of quantifying the occupied space, at the elementary and secondary school levels. The epistemology of quantification requests the full recognition of the problems which students can (or must) be faced with:

- compare two volumes (or more).
- evaluate one volume with the help of another volume (smaller or bigger).
- approximate one volume.
- evaluate one volume by combining information about lengths, areas, angles ...
- compare volumes in relationship with the comparison of lengths and areas.
- coordinate unit-systems.

and so on.

Young children are faced with some of these problems at a very young age: compare the amount of drink in two glasses; compare two containers, eventually by counting the number of cups contained in each of them; realize that the number found depends on the size of the cup; approximate measures that cannot be represented by whole numbers ...

These problems do not imply any other cognitive operations but those conveyed by a unidimensional conception of volume: volume is a measure of space contained in containers and it can be compared, added and subtracted. the addition axiom of the theory of measure is essential.

$$\text{Measure } (A \cup B) = \text{Measure } (A) + \text{measure } (B)$$
provided A and B have no common part

But it does not imply any relationship between volume and other spatial measures such as lengths and areas, except that there are some valid inferences like the following one:

If the shape of container A and container B is the same, height in A > height in B $\Longrightarrow$ quantity in A > quantity in B.

Children don't need anything but a unidimensional conception of volume, to say that:

- container X is bigger than container Y.
- container T is just as big as container X and container Y together.
- container W is 3 times as big as container Y.

- container T is two times and a half as big as container X, plus a little.
- the measures (numbers) attached to these containers are not the same if you take a small cup or a larger one as a unit.

All these tasks can be studied with 5 to 11 year-old children, either in individual interviews or in the class-room. This study shows that the concept of volume cannot be characterized by the conservation of volume experiments only. Beside the important invariant that Piaget has characterized, one finds several relational invariants like those described above. The addition axiom for instance is a decisive step in the understanding of the unidimensional conception.

But this conception does not say much about the tridimensional conception of volume, as a product of measures: it does not say anything about the calculation of the volume of parallelepipeds, prisms, pyramids or spheres, nor about the calculation of the area of the base knowing the volume and the height of a prism, nor about the measure of homothetic or similar shapes, nor about the relationship between the change of units of length and the change of units of volume.

There are situations (for instance the two-aquarium problem) in which you have to use these properties of the concept. The cognitive analysis of these situations is essential, so as to make the mathematical theory and the formulas really functional.

A tridimensional conception of volume is undoubtedly required to understand the following theorem: if you multiply by 2 the linear dimensions of a solid, the volume is multiplied by 2^3.

2. Contents of knowledge and natural language

I will come again to the comparative difficulty of problems and to the necessity of analyzing the physical and mathematical contents of the tasks. But I would like to raise now the problem of natural language, as I may have surprised you by mentioning natural language in the title of my conference. I know I have surprised Lauren Resnick anyway. What do I mean by a content-oriented approach concerning natural language?

Language, when used in a situation, especially in problem-solving, has two correlative functions: communication and representation. The communication function raises the question: who speaks to whom? The representation function: what is the speaker talking about? Other questions concern the nature of the task and the circumstances. None of these questions is inde-

pendent of the others. Schneuwly and Bronckard (1983) have made the distinction between "theoretical discourses" and "discourses in situation", by arguing that some discourses are independent of the situations and do not contain the linguistic markers that refer to the hic et nunc situation: persons and objects, location, time, essentially. These markers such as pronouns and adverbs (he, this, here, before, now, after, ...) are very frequent in "discourses in situation" and make some of them totally ununderstandable for persons who don't know the situation. This is the case when you listen to a tape, for instance. They also describe a third category of texts, narrative discourses, which describe the situation referred to.

When you observe students solving problems or asking questions about a situation, it is most likely that their verbal productions will be of the "discourse in situation" kind, although they may occasionally produce theoretical discourses (independent of the situation) and narrative discourses (referring to other situations).

As I want to illustrate the importance of a content-oriented approach, I need to go a little further in the analysis of the representation function. Language can be either a help in conceptualizing a situation or a help in executing a sequence of actions, especially mental actions; it can also be both at the same time. This double function of language in problem-solving is not specific of natural language, it is also the sort of assistance you can get from a diagram, a picture, a table, an equation or a graph. It is different to put a problem into equations or put it into words, and to manipulate algebraic symbols or reason with words; but the function of representation concerns both algebra and natural language.

The "4 times more" expression that I have mentioned before can be used by a student when he explains how to find the solution to another student or to the teacher. I have also stressed the fact that it represents a scalar relationship and not a function. I would like to take a few other examples.

2.1. How do we express the unit value

In a proportion situation, there must be at least one piece of information relating a magnitude of the first type to the corresponding magnitude of the second type.

6 bottles of beer cost 30 francs.	$f(6) = 30$
in $3\frac{1}{2}$ hours I drive 380 kilometers.	$f(3,5) = 380$ km
the weight of 5 liters of oil is 4 kilogrammes.	$f(5) = 4$

But when we express the unit value f (1) = a, we very rarely use the number "one". I have listed some expressions that we use in French.

Les gâteaux coûtent 4 francs | **chacun**
chaque
pièce
la pièce

Chaque gâteau coûte 4 francs

Un gâteau coûte 4 francs | **pièce**
la pièce

Papa a donné 2 bonbons à **chaque** enfant

Papa a acheté 2 gâteaux | **par** enfant
pour chaque enfant

Maman roule à 120 km **à l'**heure
km-heure (juxtaposition)

Un client **sur** trois prend de la salade

L'huile coûte 15 francs **le** litre.

Many of the above expressions do not refer explicitly the unit value to the unit. This may be one of the reasons why the simple multiplication problem in example 4 is not viewed as a four-term relationship but only as a three-term relationship.

2.2. How can we formulate questions unambiguously? How can we generate questions?

In an experiment with 16-year-old students, in a professional section (technical drawing for house-building) the students were given a rather large number of data and information concerning the size of the foundations of different fixed types of houses, the proportion of cement, sand and gravel needed to make concrete, and also the prices of cement, sand and gravel. A firm was supposed to submit a proposal for the foundations. No question was asked. The students were required to formulate their own questions, individually at first, and then a group of three or four students would list their set of questions as a group (which means that they

would have to come to an agreement upon the list and the formulation). Finally all the questions asked by the different groups would be written on the board.

What did we observe?

Some general features that you can imagine easily:

- Many students started with questions like "the price of all the houses" and then, by a retroactive means-end analysis, arrived at questions like the price of the cement for one house of a certain type, and the quantity of cement necessary for it.

- Some students confused the concept of volume with the concept of weight and with the more general and vague concept of quantity. As the information concerning concrete is given in terms of volume, whereas the information concerning cement, sand and gravel is given in terms of weight, it takes some time for students to arrive at unambiguous wordings. Different words for different concepts is a help, provided you understand what they discriminate.

A less expected observation was that the students changed the wording of the questions when they came to the stage of "the official list of questions of the group". To find a socially-acceptable wording is in itself a problem to be solved. I will give two examples.

- Formulation of a universal quantifier

"pour les semelles de toutes les maisons" - first sentence

"pour toutes les semelles" - shorter sentence and change of the location of the quantifier.

"nécessaire à la construction des semelles" - more explicit sentence but loss of the quantifier.

"nécessaire à la construction de l'ensemble des semelles - more sophisticated expression of the quantifier.

- Formulation of a difficult concept

A student suddenly proposes to express the foundation "sous la forme du linéaire". By this, he means that when the section (width and depth) is constant, volume can be expressed as a length: this is the way the informa-

tion is given in the text and students have larned that. So he starts:

"la longueur totale ... la longueur totale ... des semelles ... pour toutes ... de longueur ... de semelles"

You see how embarrassed he is. He also uses twice the quantifier.
The final expression is:
"La longueur totale des semelles pour l'ensemble de toutes les maisons" (three times the quantifier).

Some students fail totally to produce intelligible questions by using such expressions as "sous la forme du linéaire", "avec la méthode du linéaire", "en mètres linéaires". This difficulty is due to the difficulty of the concept itself which relies upon a rather abstract theorem:

> information concerning the volume can be given in terms of length, because the volume of the foundation is proportional to the length when the section is held constant.

But one also sees that formulation can be approached as a problem to be solved.

- Suppression of redundancy and use of a pronoun

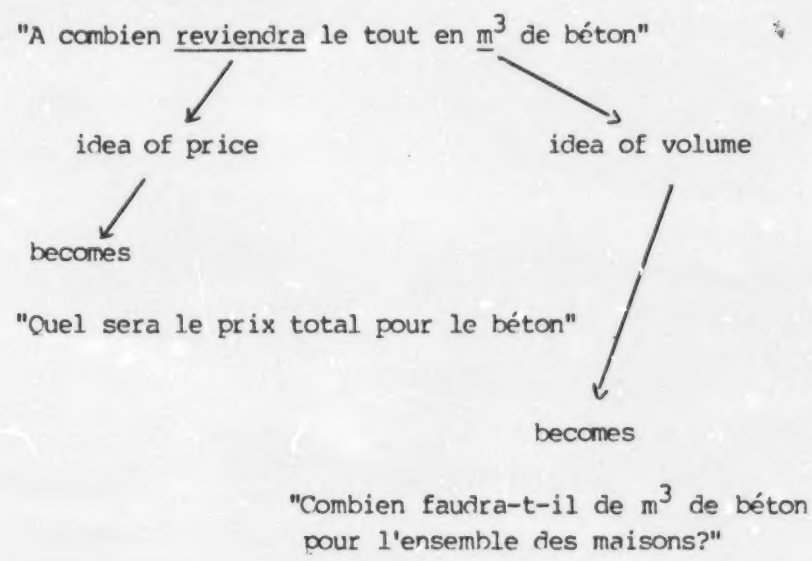
"A combien sera évalué le coût de ..." (two words for the same concept).

and later "Combien coûtera le prix ..." (two words again) is corrected into "quel sera le prix?"

and another student adds "quel sera le prix de toutes les semelles" (détermination)

another student says "Quel en sera le prix" ("de toutes les semelles" has already been expressed in the preceding question).

- Two questions in one; become two


"A combien reviendra le tout en m^3 de béton"
idea of price
idea of volume
becomes
"Quel sera le prix total pour le béton"
becomes
"Combien faudra-t-il de m^3 de béton pour l'ensemble des maisons?"

- Extraction of an invariant and recognition of the possibility to substitute paradigmatic expressions

Later on, some students recognize the invariant "Quelle quantité de ciment faut-il pour" and they refer it successively to different types of houses and different numbers of houses. Immediately afterwards, one student recognizes the fact that one can also vary the nature of the quantity:

"Combien de tonnes de sable ..."
"Combien de tonnes de gravillons ..."

and he comments:

"On peut faire varier les différentes valeurs .. par exemple ... de la phrase ..." which is a good spontaneous metalinguistic comment.

This analysis is a help for the next task, in which students are asked to place data and questions in a double-entry table like this one:

	longueur de semelle	weight of cement	weight of sand	weight of gravel	volume of concrete	price of cement	price of sand	price of gravel
1 F_4								
1 F_5								
1 F_6								
12 F_4								
.								
.								
.								
.								

This table exhibits the important fact that each question refers to one kind of magnitude and to a certain number of houses of a certain type.

But paradoxically, the recognition of a sentence invariant may also lead to wrong formulations. For intance the question:

"Calculer la masse volumique d'une semelle en béton"

is distributed over different types and numbers of houses. This is of course meaningless as the volumic mass of concrete is invariant. It takes students some time to arrive at the idea that "it is always the same", "it is always constant".

I have given these examples with some details because they illustrate the interaction between language and thought and the necessity for us to refer the analysis of verbal productions to the contents of knowledge.

Two more examples with younger children will illustrate the double function of language as an assistance in conceptualizing a new problem and in controlling action.

2.3. How language can help controlling action and conceptualizing a new situation

The information given to Charlotte (2nd grade) is:

"Véronique had bought 24 postcards. She has written to her friends. She has now 11 postcards left."

No question is asked. Charlotte is supposed to formulate a question and then give the answer. She can use chips to solve the problem.

Charlotte

Ch. : **On pourrait se demander combien elle en a utilisé.**
(Ch. prend les jetons et dit:) **je vais en prendre 24 - 1, 2, 3 ... 24** (compte tout fort) **et 24 ça y est. Alors je vais en garder 11 maintenant, je vais en prendre 11, 5 et 6 alors ça fait ... ça me fait 10 déjà, alors il m'en manque une, en voilà un ... alors là y' en avait 11, je vérifie parce que 1, 2, 3, 4 ... 10, 11** (compte tout fort). **Alors là j'en ai 11, j'écris là** (montre cahier). **Il m'en reste combien? 2, 4, 6, 8, 10, 12, 14** (groupe les jetons par 2 et compte tout fort) **Y'en a 14.**

Exp: Tu es sûre?

1, 2, 3, 4 ... 13 (recompte tout fort) **Ah, 13.**

Exp: Qu'est-ce que tu peux dire?

C'est les 13 cartes postales qu'elle a envoyées.

You can notice that Charlotte asks the relevant questions without any hesitation. Then she uses chips to calculate. The control by language appears several times, in different ways.

- she explicits what she is doing
- she checks
- she explicits the first step of the procedure
- she writes it
- and she counts again to find the result (her first answer is wrong).

The specific parameters of "enunciation" are visible.

- pronouns: je, me
  ça me plaît, il m'en manque

- temporal and spatial deictics: maintenant, là

- designation operators: ça, ça me plaît

They are typical of "discourses in situation".

The same information is given to Pascale (2nd grade)

Pasc.: starts with an addition before formulating any question.

finds 24
+ 11

35

Exp.: C'est quoi ces 35?

Pasc.: **que ... 24 plus 11, on peut ...**

Exp.: Est-ce que les 35, c'est ce qu'elle a envoyé?

Pasc.: **Non**

Exp.: Pourquoi?

Pasc.: **Parce que 24 ça c'était avant et puis 11 c'était après.**

Exp.: Alors, qu'est-ce qu'il faut faire?

Pasc.: **Je crois bien qu'il faut que je prende les jetons? et que je peux barrer. Faut pas trop faire de ratures quand même.** (Pasc. prend les jetons). **Je prends 24 jetons.** (Les prend dans la main puis les pose et compte jusqu'à 24).

Exp.: Qu'est-ce que tu vas faire?

Pasc.: **Je vais enlever des jetons jusqu'à 11.** (En fait, Pasc. enlève 11 jetons)

Exp.: Et maintenant, qu'est-ce que tu fais?

Pasc.: **Et maintenant je vais marquer qu'y en avait ...**

Exp.: Tu en as pris combien de jetons?

Pasc.: **J'en ai pris 24.**

Exp.: Qu'est-ce que tu as fait?

Pasc. **J'ai enlevé ... j'ai pris les jetons j'ai enlevé ... mes 11 jetons et là il m'en reste** (compte tout bas 1, 2 ... 12) **et là il m'en reste 12.**

Exp.: Je crois bien qu'il y a une petite erreur.
(pasc. recompte les 2 tas).

Pasc.: **13.**

Exp.: Qu'est-ce que tu peux dire? Quelle phrase pourrais-tu faire?

Pasc.: **Peut-être ça serait qu'elle en a envoyé 13 et il lui en reste 11.**

Exp.: Tu penses que c'est ça?

Pasc.: **Oui.**

Exp.: Tu saurais écrire l'opération?

Pasc.: **Non je saurais pas.**

Pascale never formulates the question but she uses both language and chips to elaborate a correct representation of the problem and the solution. Her last sentence shows that her final representation is quite good.

These protocols have been collected by Danièle Morange for her thesis. One of the important findings is that most children that can formulate the relevant question can also find the answer. The answer and the question seem to go together.

3. Conclusion

I have raised several theoretical problems :
- The need for a reference to situations and the importance of a complete analysis of the relationships involved in the task.
- The concept of "theorem-in-action" enables us to characterize a competence in terms that are compatible with the mathematical and physical contents.
- Finally the components of natural language and the other symbolic systems play an important part in the process of conceptualizing a new situation and accompanying action to handle it:

These considerations have led me to a comprehensive definition of what a concept consists of. It is a triplet of three sets.

$$\text{concept} = (S, I, \mathcal{S})$$

S : the set of situations that make the concept meaningful
I : the set of invariants (or theorems-in-action) that characterize the variety of competences of students. These invariants are properties of the concept.
$\mathcal{S}$: the set of symbolic representations that can be used to represent these properties and the situations.

In other words I would call S the referent, I the signified, $\mathcal{S}$ the signifier.

The core of cognition lies in the operational invariants that students gradually discover or appropriate. I call them theorems-in-action because, after all, they are theorems. But there is a large variety of them, much larger than what we have identified up to now. The variety of situations is still larger and the handscape is made more complex by the fact that :
not only does a concept refer to a variety of situations, but each single situation cannot usually be analysed with only one concept. For multiplicative structures for instance, we need the concepts of quantity, scalar, multiplication, division, linear and n-linear function, dimensional analysis, fraction, ratio...
Because a concept refers to a large set of situations and because there are a variety of invariants and a variety of symbolic expressions for these invariants, we need to study rather extensive conceptual fields.

The concept of conceptual field aims at cutting out fair-sized objects for research on the development, the learning and the teaching of concepts and competences. It relies upon a content-oriented approach and not upon a logical, a linguistic or a structural approach. A reliable description of the development of knowledge in students'minds, cannot avoid the contents of knowledge.

A conceptual field consists, first of all, in a set of situations. We must classify them and analyse the relationships that constitute the core of each class of problems. We must experiment with them at different levels, under different conditions, for different values of the variables, and then try to understand the variety of procedures and behaviors of students. We must also experiment different ways of representing them, analyse what is well represented in a certain symbolic system, and what is not, and then understand the words, pictures and symbols used by students.

For instance I have studied extensively the concept of number line as a representation of numbers and differences, of dates and durations. The results show that the understanding and the use of the number line is also, like language, a big problem to be solved, which has a lot to do with the analysis of additive structures and the analysis of space. It is not mainly a problem of syntax, it is a conceptual problem. The same is true for other symbolic systems like equations and diagrams, tables and graphs.

Conceptions of students depend on situations they have met. Primitive conceptions of addition and subtraction, multiplication and division, are shaped by the first situations mastered by students.

But we must also analyse how students modify these initial conceptions, by facing new situations, identifying similarities and differences, widening the scope of the first operations and eventually rejecting erroneous conceptions. During that process, which covers a very long period of time (over twelve years for additive and multiplicative structures), there are many metaphoric processes, and a few conceptual revolutions ; also many misunderstandings. It is only a detailed description that can really enable teachers to interpret what students do and to find the explanations (also the questions and situations) that may help them.

The importance of the contents is easily recognised for physics, chemistry, biology, economics... Nobody is surprised to read in a research-paper that the primitive conceptions of the electric current raise different epistemological obstacles from those raised by the primitive conceptions of movement, and force.

For instance the main epistemological difficulty for electric current is that it is viewed as the circulation of some fluid in a house equipment. The more you go along the circuit, the less electricity you have and what happens further does not depend upon what happens before (Closset 1983). This conception contradicts electrocinetic relationships.

On the other hand the formula for the potential

$$U = RI$$

is very similar to the formula for the quantity of movement

$$Q = MV$$

Both formulas rely upon the analysis I have presented before : the first variable is proportional to each of the other two separately.

I have identified different categories of divisions in the simple proportion scheme and the double proportion scheme and made an experiment

with different domains of physics.

We found that the competences of our 16 year-old students were, after all, fairly good. About 60% of them could find the solution of problems requiring either two divisions, or one division and one multiplication, when the domain was conceptually easy.

The success rate fell down to 10 % for the massic heat, which is proportional to the mass and to the difference of temperature; although the students had studied the domain.

So the contents is essential in physics.

But I would like to add two remarks.

The mathematical model of double proportion is essential in the conceptualization of electrokinetics, massic heat, and many other physical concepts. So the problem is not purely physical, it is also mathematical.

And finally the conceptual difficulties of additive structures and multiplicative structures are very different. The main difficulties in multiplicative structures lie in the dimensional properties of the quantities and operators involved, and in the dependance/independance of variables ; they also come from the primitive conception that students have of fractions and ratios. Whereas the main difficulties in additive structures, lie in the temporal aspects, especially the temporal transformations : inversion, combination and decombination of transformations. Mathematics is concerned by a contents-oriented approach.

My final conclusion will be of an epistemological nature.

The meaning of knowledge is conveyed by practical and theoretical problems to be solved, provided they are real problems for students. This is of course a functionnalist's point of view. It has probably some limits. But for the time being, it would be profitable, for research on education, to try to answer the following question:
"to which problem or problems, does a new concept, a new property of a concept, a new procedure, a new representation or a new formulation bring a solution or eventually a better one ?"

This analysis cannot be done without a careful and profound analysis of the contents of knowledge.

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